Brownian Motion and Stochastic Calculus
Dylan Possamaï

Assignment 1—solutions

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a filtration $\mathbb{F}$.

On hitting times of closed and open sets

Let E be a finite-dimensional Euclidean space, and let X be an E-valued and $\mathbb{F}$-adapted process.

1) Prove that if $G \subset E$ is open and X has right- or left-continuous trajectories, then $\rho_{X,G}$ and $\tau_{X,G}$ are $\mathbb{F}$–optional times (see Definition 1.3.3 in the lecture notes)

   Let us fix some $t > 0$. Then

   $$\{\tau_{X,G} < t\} = \bigcup_{0 \leq s < t} \{X_s \in G\} = \bigcup_{s \in [0,t) \cap \mathbb{Q}} \{X_s \in G\} \in \mathcal{F}_t,\ \{\rho_{X,G} < t\} = \bigcup_{0 < s < t} \{X_s \in G\} = \bigcup_{s \in (0,t) \cap \mathbb{Q}} \{X_s \in G\} \in \mathcal{F}_t,$$

   where we used both the continuity of X and the fact that G is open to reduce the intersection to rational times only.

2) Show that if $G \subset E$ is closed and X has continuous trajectories, then $\tau_{X,G}$ is an $\mathbb{F}$–stopping time.

   Since G is closed, we can define the distance function to G by $E \ni x \overset{d(\cdot,G)}{\longmapsto} \inf_{y \in G} \|x - y\|_E$. For any positive integer n, we define then

   $$G_n := \big\{x \in E : d(x, G) < 1/n\big\}.$$

   Since the distance function is continuous, G_n is open for each $n \in \mathbb{N}^\star$, and $\tau_{X,G_n} < \tau_{X,G}$ on the set $\{\tau_{X,G} \in (0, +\infty)\}$. Indeed, since X is right-continuous and G is closed, then $X_{\tau_{X,G}} \in G$ on $\{\tau_{X,G} < +\infty\}$, and the left-continuity of X implies that $\tau_{X,G_n} < \tau_{X,G}$ on the set $\{\tau_{X,G} \in (0, +\infty)\}$. On the set $\{\tau_{X,G} \in (0, +\infty)\}$, the sequence $(\tau_{X,G_n})_{n \in \mathbb{N}^\star}$ is clearly increasing, and therefore converges to some $\tau(\omega) \leq \tau_{X,G}(\omega)$ for every $\omega \in \Omega$. The left-continuity of X implies that necessarily, $\tau(\omega) = \tau_{X,G}(\omega)$ for every $\omega \in \Omega$. Then, because G is closed and X is right-continuous

   $$\{\tau_{X,G} \leq 0\} = \{\tau_{X,G} = 0\} = \{X_0 \in G\} \in \mathcal{F}_0,$$

   and for any $t > 0$

   $$\{\tau_{X,G} \leq t\} = \{\tau_{X,G} = 0\} \cup \{\tau_{X,G} \in (0, t]\} = \bigcap_{n \in \mathbb{N}^\star} \{\tau_{X,G_n} < t\},$$

   which belongs to $\mathcal{F}_t$ by 1), since G_n is open.

3) Show that if $G \subset E$ is closed, and X is càdlàg then $\theta_{X,G}$ is an $\mathbb{F}$–stopping time (see Definition 1.3.5 in the lecture notes).

   The proof is similar to 2). Simply notice that for any $t \geq 0$

   $$\{\theta_{X,G} \leq t\} = \Big\{\inf\big\{d(X_r, G) : r \in \mathbb{Q} \cap [0, t]\big\} = 0\Big\} \cup \{X_t \in G\}.$$

Properties of stopping times

1) Show that if τ is an $\mathbb{F}$–optional time, and $\theta \in (0, +\infty)$, then $\tau + \theta$ is an $\mathbb{F}$–stopping time.

Fix some $t \geq 0$, we have

$$\{\tau + \theta \leq t\} = \begin{cases} \emptyset, \text{ if } t < \theta, \\ \{\tau \leq t - \theta\}, \text{ otherwise.} \end{cases}$$

Hence, since τ is an $\mathbb{F}$–optional time, se deduce that $\{\tau + \theta \leq t\} \in \mathcal{F}_{\max\{0, t-\theta\}+} \subset \mathcal{F}_t$, which implies the desired result.

2) Show that if τ and ρ are $\mathbb{F}$–stopping times, so are $\tau \wedge \rho$, $\tau \vee \rho$ and $\tau + \rho$.

Fix again some $t \geq 0$. We have

$$\{\tau \wedge \rho \leq t\} = \{\tau \leq t\} \bigcup \{\rho \leq t\} \in \mathcal{F}_t, \ \{\tau \vee \rho \leq t\} = \{\tau \leq t\} \bigcap \{\rho \leq t\} \in \mathcal{F}_t,$$

since τ and ρ are $\mathbb{F}$–stopping times.

Next, we have

$$\{\tau + \rho > t\} = \big(\{\tau = 0\} \cap \{\rho > t\}\big) \bigcup \big(\{\tau > t\} \cap \{\rho = 0\}\big) \bigcup \big(\{\tau \geq t\} \cap \{\rho > 0\}\big) \bigcup \big(\{0 < \tau < t\} \cap \{\tau + \rho > t\}\big).$$

The first three events are obviously in $\mathcal{F}_t$ since τ and ρ are $\mathbb{F}$–stopping times (which implies that for any $t \geq 0$ all the sets $\{\psi \leq t\}$, $\{\psi > t\}$, $\{\psi = t\}$, $\{\psi \geq t\}$, for $\psi = \tau, \rho$, are all in $\mathcal{F}_t$). Finally, the fourth one can be rewritten as

$$\{0 < \tau < t\} \cap \{\tau + \rho > t\} = \bigcup_{r \in (0,t) \cap \mathbb{Q}} \{r < \tau < t\} \cap \{\rho > t - r\} \in \mathcal{F}_t.$$

This proves that $\{\tau + \rho > t\}$, and thus $\{\tau + \rho \leq t\}$, belongs to $\mathcal{F}_t$.

3) Show that if τ and ρ are $\mathbb{F}$–optional times, then $\tau + \rho$ is also an $\mathbb{F}$–optional time. It is moreover an $\mathbb{F}$–stopping time if either τ and ρ are positive, or if $\tau > 0$ and τ is an $\mathbb{F}$–stopping time.

Exactly as above, we can write for any $r \geq 0$

$$\{\tau + \rho \geq t\} = \big(\{\tau = 0\} \cap \{\rho \geq t\}\big) \bigcup \big(\{\tau \geq t\} \cap \{\rho = 0\}\big) \bigcup \big(\{\tau \geq t\} \cap \{\rho > 0\}\big) \bigcup \big(\{0 < \tau < t\} \cap \{\tau + \rho \geq t\}\big).$$

The first three events are obviously in $\mathcal{F}_t$ since τ and ρ are $\mathbb{F}$–optional times (which implies that for any $t \geq 0$ all the sets $\{\psi < t\}$, $\{\psi \geq t\}$, for $\psi = \tau, \rho$, are all in $\mathcal{F}_t$). Finally, the fourth one can be rewritten as

$$\{0 < \tau < t\} \cap \{\tau + \rho > t\} = \bigcup_{r \in (0,t) \cap \mathbb{Q}} \{r < \tau < t\} \cap \{\rho \geq t - r\} \in \mathcal{F}_t.$$

This proves that $\{\tau + \rho \geq t\}$, and thus $\{\tau + \rho < t\}$, belongs to $\mathcal{F}_t$.

If now both τ and ρ are positive, we have

$$\{\tau + \rho > t\} = \{\tau \geq t\} \bigcup \big(\{\tau < t\} \cap \{\tau + \rho > t\}\big).$$

The first event is in $\mathcal{F}_t$ because τ is an $\mathbb{F}$–optional time, while the second one can be rewritten

$$\{\tau < t\} \cap \{\tau + \rho > t\} = \bigcup_{r \in (0,t) \cap \mathbb{Q}} \{r < \tau < t\} \cap \{\rho \geq t - r\} \in \mathcal{F}_t.$$

This proves that $\tau + \rho$ is then an $\mathbb{F}$–stopping time.

Finally, let us assume that τ is a positive $\mathbb{F}$–stopping time. Then we have

$$\{\tau + \rho > t\} = \big(\{\tau > t\} \cap \{\rho = 0\}\big) \bigcup \big(\{\tau \geq t\} \cap \{\rho > 0\}\big) \bigcup \big(\{\tau < t\} \cap \{\tau + \rho > t\}\big),$$

and we can conclude as before using the fact that

$$\{\tau < t\} \cap \{\tau + \rho > t\} = \bigcup_{r \in (0,t) \cap \mathbb{Q}} \{r < \tau < t\} \cap \{\rho \geq t - r\} \in \mathcal{F}_t.$$

4) Let $(\tau_n)_{n \in \mathbb{N}}$ be a sequence of $\mathbb{F}$–optional times. Show that the following four random times are $\mathbb{F}$–optional times

$$\sup_{n \in \mathbb{N}} \tau_n, \ \inf_{n \in \mathbb{N}} \tau_n, \ \overline{\lim_{n \to +\infty}} \tau_n, \ \underline{\lim_{n \to +\infty}} \tau_n.$$

Furthermore, if the $(\tau_n)_{n \in \mathbb{N}}$ are actually $\mathbb{F}$–stopping times, show that $\sup_{n \in \mathbb{N}} \tau_n$ is an $\mathbb{F}$–stopping time too.

Notice that for any $t \geq 0$, using the fact that all the $(\tau_n)_{n \in \mathbb{N}}$ are $\mathbb{F}$–optional times

$$\left\{ \sup_{n \in \mathbb{N}} \tau_n < t \right\} = \bigcup_{k \in \mathbb{N}^\star} \bigcap_{n \in \mathbb{N}} \{\tau_n < t - 1/k\} \in \mathcal{F}_t, \ \left\{ \inf_{n \in \mathbb{N}} \tau_n < t \right\} = \bigcup_{n \in \mathbb{N}} \{\tau_n < t\} \in \mathcal{F}_t,$$

which proves the first two required results. For the last two ones, it suffices to recall that by definition

$$\overline{\lim_{n \to +\infty}} \tau_n = \inf_{n \in \mathbb{N}} \sup_{k \geq n} \tau_k, \ \underline{\lim_{n \to +\infty}} \tau_n = \sup_{n \in \mathbb{N}} \inf_{k \geq n} \tau_k,$$

and to use what we just proved.

Finally, if the $(\tau_n)_{n \in \mathbb{N}}$ are $\mathbb{F}$–stopping times, we have

$$\left\{ \sup_{n \in \mathbb{N}} \tau_n \leq t \right\} = \bigcap_{n \in \mathbb{N}} \{\tau_n \leq t\} \in \mathcal{F}_t.$$

Hitting times and completeness of $\mathbb{F}$

Let X be an $\mathbb{R}$-valued and right-continuous process. The goal of this exercise is to show that for any $M \in \mathbb{R}$, the hitting time $\tau_{X,[M,+\infty)}$ is an $\mathbb{F}$–stopping time when $\mathbb{F}$ is $\mathbb{P}$-complete.

1) Given any $\mathbb{F}$–stopping time σ which is below $\tau_{X,[M,+\infty)}$, define

$$\sigma^+ := \inf \left\{ t \geq \sigma \colon \sup_{\sigma \leq u \leq t} X_u \geq M \right\}.$$

Show that $\sigma \leq \sigma^+ \leq \tau_{X,[M,+\infty)}$, that σ^+ is still an $\mathbb{F}$–stopping time, and that σ^+ is strictly greater than σ whenever $\sigma < \tau_{X,[M,+\infty)}$.

It is immediate that $\sigma \leq \sigma^+ \leq \tau_{X,[M,+\infty)}$, and we claim that σ^+ is still an $\mathbb{F}$–stopping time. Indeed, σ^+ is less than or equal to a non-negative time t precisely when, for each $\varepsilon > 0$, there is a time s in-between σ and t such that $X_s > M - \varepsilon$. By right-continuity, it is enough to restrict to rational multiples of t, which means that

$$\{\sigma^+ \leq t\} = \bigcap_{n=1}^{\infty} \bigcup_{q \in \mathbb{Q} \cap [0,1]} \big(\{\sigma \leq qt\} \cap \{X_{qt} > M - 1/n\} \big) \in \mathcal{F}_t.$$

Furthermore, using right continuity again, σ^+ will be strictly greater than σ whenever $\sigma < \tau_{X,[M,+\infty)}$.

2) Let $\mathcal{T}_{0,\tau_{X,[M,+\infty)}}$ consist of the set of all $\mathbb{F}$–stopping times σ satisfying $\sigma \leq \tau_{X,[M,+\infty)}$, and define

$$\sigma_\infty := \operatorname{essup}^{\mathbb{P}} \mathcal{T}_{0,\tau_{X,[M,+\infty)}}.$$

Prove that the family $\mathcal{T}_{0,\tau_{X,[M,+\infty)}}$ is upward directed and deduce that there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ valued in $\mathcal{T}_{0,\tau_{X,[M,+\infty)}}$ such that

$$\sigma_\infty = \sup_{n\in\mathbb{N}} \tau_n.$$

The family $\mathcal{T}_{0,\tau_{X,[M,+\infty)}}$ is upward directed, since the maximum of two $\mathbb{F}$–stopping times below $\tau_{X,[M,+\infty)}$ is itself an $\mathbb{F}$–stopping time below $\tau_{X,[M,+\infty)}$ by the previous exercise. Therefore, there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ valued in $\mathcal{T}_{0,\tau_{X,[M,+\infty)}}$ such that

$$\sigma_\infty = \sup_{n\in\mathbb{N}} \tau_n.$$

3) Show that $\sigma_\infty \in \mathcal{T}_{0,\tau_{X,[M,+\infty)}}$ and that $\sigma_\infty^+ = \sigma_\infty$, $\mathbb{P}$–a.s.

Again by the previous exercise, we have that $\sigma_\infty \in \mathcal{T}_{0,\tau_{X,[M,+\infty)}}$, see question 4). The $\mathbb{F}$–stopping time σ_∞^+ defined as above satisfies therefore $\sigma_\infty \leq \sigma_\infty^+ \leq \tau_{X,[M,+\infty)}$ and is therefore also in $\mathcal{T}_{0,\tau_{X,[M,+\infty)}}$. From the definition of the $\mathbb{P}$–essential supremum, this implies that $\sigma_\infty^+ \leq \sigma_\infty$ and, therefore, $\sigma_\infty^+ = \sigma_\infty$ with $\mathbb{P}$–probability one.

4) Deduce that $\sigma_\infty = \tau_{X,[M,+\infty)}$ and conclude.

As mentioned in 1), $\sigma_\infty^+ > \sigma_\infty$ whenever $\sigma_\infty < \tau_{X,[M,+\infty)}$, which therefore has zero $\mathbb{P}$–probability, and we conclude using the $\mathbb{P}$–completeness of $\mathbb{F}$.